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THE FLUX HOMOMORPHISM AND CENTRAL EXTENSIONS OF DIFFEOMORPHISM GROUPS

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Abstract

Let D be a closed unit disk in dimension two and G_{rel} the group of symplectomorphisms on D preserving the origin and the boundary ∂D pointwise. We consider the flux homomorphism on G_{rel} and construct a central $\mathbb{R}$-extension called the flux extension. We determine the Euler class of this extension and investigate the relation among the extension, the group 2-cocycle defined by Ismagilov, Losik, and Michor, and the Calabi invariant of D.

1. Introduction

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be a closed unit disk with a symplectic form $\omega = dx \wedge dy$. Put $o = (0, 0) \in D$. Let us denote by α^g the pullback of a differential form α by a diffeomorphism g. Denote by $G = \{g \in \mathrm{Diff}(D) \mid \omega^g = \omega, g(o) = o\}$ the group consisting of symplectomorphisms on D that preserve the origin and set $G_{\mathrm{rel}} = \{g \in G \mid g|_{\partial D} = \mathrm{id}_{\partial D}\}$. Then there is the following exact sequence:

$$1 \longrightarrow G_{\mathrm{rel}} \longrightarrow G \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

where $\mathrm{Diff}_+(S^1)$ is the group of orientation preserving diffeomorphisms on the unit circle $S^1 = \partial D$. On the group G_{rel}, there is an $\mathbb{R}$-valued homomorphism

$$\mathrm{Flux}_{\mathbb{R}} : G_{\mathrm{rel}} \to \mathbb{R}\, ; g \mapsto \int_\gamma \eta^g - \eta$$

called the flux homomorphism, where γ is a path from the origin to the boundary of D and η is a 1-form satisfying $\omega = d\eta$. If we take η as in (3.3), the homomorphism $\mathrm{Flux}_{\mathbb{R}}$ seems to be the pairing of the ordinary flux homomorphism $\mathrm{Flux} : G_{\mathrm{rel}} \to H^1(D, \partial D \cup \{o\}; \mathbb{R}); g \mapsto [\eta^g - \eta]$ and the generator of the singular homology $H_1(D, \partial D \cup \{o\}; \mathbb{Z})$. Although, for an arbitrary η, the closed form $\eta^g - \eta$ is not necessary to define the relative cohomology class (and thus the ordinary flux homomorphism is not well-defined in this case), the integral $\int_\gamma \eta^g - \eta$ remains meaningful. Moreover, the integral is also independent of the choice of η.

Dividing the above exact sequence by the kernel $K = \mathrm{Ker}\,\mathrm{Flux}_{\mathbb{R}}$, we obtain the following central $\mathbb{R}$-extension:

$$0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

which we call the flux extension. Denote by $e(G/K)$ the Euler class of the flux extension, namely the cohomology class in $H^2(\mathrm{Diff}_+(S^1); \mathbb{R})$ corresponding to the given extension.

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There is another cohomology class $e_{\mathbb{R}}$ in $H^2(\mathrm{Diff}_+(S^1);\mathbb{R})$, which is the class corresponding to a universal covering space of $\mathrm{Diff}_+(S^1)$. The following theorem clarifies the relation between $e(G/K)$ and $e_{\mathbb{R}}$:

Theorem A (Theorem 3.10)**.** *The class $e(G/K)$ is equal to $\pi e_{\mathbb{R}}$.*

For a symplectic manifold M endowed with an exact symplectic form ω, there is a 2-cocycle C_{η,x_0} on $\mathrm{Symp}(M)$ for some η satisfying $\omega = d\eta$ and $x_0 \in M$ defined by Ismagilov, Losik, and Michor [4], which we call the ILM cocycle. Let us consider the ILM cocycle C_{η,x_0} on $H = \mathrm{Symp}(D)$ for $\eta = (xdy - ydx)/2$ and $x_0 = (1,0) \in \partial D$. It turns out that the ILM cocycle is related to the flux homomorphism. Let $\tau : G \to \mathbb{R}$ be the map defined in the same way as for the flux homomorphism, that is, $\tau(g) = \int_\gamma \eta^g - \eta$, where γ is defined as (3.3) in section 3. Then the ILM cocycle is equal to the coboundary $-\delta\tau$ on G.

The ILM cocycle also relates to the Calabi invariant. Set $H_{\mathrm{rel}} = \{g \in H \mid g|_{\partial D} = \mathrm{id}\}$. The Calabi invariant $\mathrm{Cal} : H_{\mathrm{rel}} \to \mathbb{R}$ is defined as

$$\mathrm{Cal}(g) = \int_D \eta \cup \delta\eta(g) = \int_D \eta^g \wedge \eta.$$

Let $\tau_0 : H \to \mathbb{R}$ be the function defined by the same formula of Cal. Then it turns out that the ILM cocycle coincides with $\delta\tau_0$. From this, we have

Theorem B (Theorem 5.7)**.** *Take η and x_0 as above. The ILM cocycle C_{η,x_0} is basic, that is, there exists a cocycle χ on* $\mathrm{Diff}_+(S^1)$ *such that $C_{\eta,x_0} = p^*\chi$ with the restriction $p : H \to$* $\mathrm{Diff}_+(S^1)$*. Furthermore, the cohomology class $[\chi]$ is equal to $\pi e_{\mathbb{R}}$.*

Furthermore, we can generalize Theorem B for arbitrary choices of $x_0 \in \partial D$ and η satisfying $\omega = d\eta$.

The present paper is organized as follows. In Section 2, we briefly recall the Euler class of a central extension. We describe a cocycle representing the Euler class in terms of a connection cochain and its curvature. In Section 3, we construct the flux extension and we prove Theorem A. In Section 4, we introduce the ILM cocycle. Finally, in Section 5, we discuss the relation between the Calabi extension and the ILM cocycle, and prove Theorem B.

2. Central extensions and the Euler class

Let Γ be a group and A a right Γ-module. We denote the action by a^g for $a \in A, g \in \Gamma$. For a non-negative integer p, a p-cochain is an arbitrary map $c : \Gamma^p \to A$, where Γ^p denotes the p-fold product group of Γ. Set $C^p(\Gamma; A) = \{c : \Gamma^p \to A \mid p\text{-cochain}\}$ and define the coboundary map $\delta : C^p(\Gamma; A) \to C^{p+1}(\Gamma; A)$ by

$$\begin{aligned}\delta c(g_1,\ldots,g_{p+1}) &= c(g_2,\ldots,g_{p+1}) \\ &\quad + \sum_{i=1}^{p}(-1)^i c(g_1,\ldots,g_i g_{i+1},\ldots,g_{p+1}) + (-1)^{p+1}c(g_1,\ldots,g_p)^{g_{p+1}}\end{aligned}$$

for $c \in C^p(\Gamma; A)$ and $g_1,\ldots,g_{p+1} \in \Gamma$. These cochains and cobundary maps give rise to a cochain complex $(C^\bullet(\Gamma, A), \delta)$ and its cohomology $H^*(\Gamma; A)$ is called the group cohomology (see [1] for more details).

If A is a trivial Γ-module, the group cohomology $H^2(\Gamma; A)$ has a description in terms of central extensions of groups. Recall that a *central A-extension of* Γ is an exact sequence of groups

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} \Gamma \longrightarrow 1$$

such that the image $i(A)$ is contained in the center of E. It is known that each equivalence class of central A-extensions corresponds to a second cohomology class in $H^2(\Gamma; A)$ (see [1]). For a central A-extension E, the corresponding cohomology class $e(E) \in H^2(\Gamma; A)$ is called the *Euler class of the central A-extension E*.

To investigate the Euler class at the cochain level, we introduce the notions called a connection cochain and its curvature. This is similar to the Chern-Weil theory defining characteristic classes of principal bundles.

Definition 2.1. Let E be a central A-extension of Γ. A 1-cochain $\tau \in C^1(E; A)$ is called a *connection cochain of E* if τ satisfies

$$\tau(ea) = \tau(e) + a$$

for any $e \in E, a \in A$. The coboundary $\delta\tau \in C^2(E; A)$ is called a *curvature of* τ.

A curvature is basic, that is, there exists a unique 2-cocycle $\sigma \in C^2(\Gamma; A)$ such that the pullback of σ to E coincides with $\delta\tau$. We call this cocycle σ a basic cocycle of $\delta\tau$. Then the cohomology class $[-\sigma]$ is equal to the Euler class $e(E)$ (see [7] for more details).

Let B be an abelian group and $\iota : A \to B$ a homomorphism. A 1-cochain $\tau_B \in C^1(E; B)$ is called a *(B-valued) connection cochain* if τ_B satisfies $\tau_B(ea) = \tau_B(e) + \iota(a)$ for every $e \in E, a \in A$. The curvature $\delta\tau_B$ is basic and we denote by $\sigma_B \in C^2(\Gamma; B)$ the basic cocycle. Then the class $[-\sigma_B]$ corresponds to the image of Euler class $e(E)$ with respect to a natural homomorphism $H^2(\Gamma; A) \to H^2(\Gamma; B)$.

Example 2.2. Let us consider the real Euler class $e_{\mathbb{R}} \in H^2(\mathrm{Diff}_+(S^1); \mathbb{R})$, which is the image of the Euler class of a central extension $0 \to \mathbb{Z} \to \widetilde{\mathrm{Diff}_+(S^1)} \to \mathrm{Diff}_+(S^1) \to 1$ under the homomorphism $H^2(\mathrm{Diff}_+(S^1); \mathbb{Z}) \to H^2(\mathrm{Diff}_+(S^1); \mathbb{R})$. For any real number $c \in \mathbb{R}$, a 1-cochain τ on $\widetilde{\mathrm{Diff}_+(S^1)}$ is defined by $\tau(\varphi) = \varphi(c)/2\pi \in \mathbb{R}$, where an element φ in $\widetilde{\mathrm{Diff}_+(S^1)}$ is considered as a diffeomorphism on $\mathbb{R}$ satisfying $\varphi(x + 2\pi) = \varphi(x) + 2\pi$ for any $x \in \mathbb{R}$. This cochain τ is a ($\mathbb{R}$-valued) connection cochain and minus the basic cocycle is

$$\chi(\mu, \nu) = -\delta\tau(\varphi, \psi) = \frac{\varphi\psi(c) - \varphi(c) - \psi(c)}{2\pi}, \tag{2.1}$$

where $\varphi, \psi \in \widetilde{\mathrm{Diff}_+(S^1)}$ are lifts of $\mu, \nu \in \mathrm{Diff}_+(S^1)$ respectively. This cochain χ is a cocycle representing the real Euler class $e_{\mathbb{R}}$.

3. The flux extension

Next we introduce another central $\mathbb{R}$-extension of $\mathrm{Diff}_+(S^1)$. Let G be the group of symplectomorphisms on D preserving the origin $o = (0, 0) \in D$:

$$G = \{g \in \mathrm{Diff}(D) \mid \omega^g = \omega, g(o) = o\},$$

where ω^g is the pullback of ω by g. We denote by G_{rel} the subgroup of G consisting of symplectomorphisms which are the identity on boundary. Then there is the following exact sequence:

$$1 \longrightarrow G_{\mathrm{rel}} \longrightarrow G \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1.$$

DEFINITION 3.1. The *flux homomorphism* $\mathrm{Flux}_{\mathbb{R}} \in C^1(G_{\mathrm{rel}}; \mathbb{R})$ is defined as

$$\mathrm{Flux}_{\mathbb{R}}(g) = -\int_\gamma \delta\eta(g) = \int_\gamma \eta^g - \eta \tag{3.1}$$

for $g \in G_{\mathrm{rel}}$.

REMARK 3.2. The flux homomorphism $\mathrm{Flux}_{\mathbb{R}}$ is independent of the choice of η because of the equality $\int_\gamma \alpha^g - \alpha = 0$ for any closed 1-form α on D. It is also independent of the choice of path γ connecting the origin o and a point on boundary because $\eta^g - \eta$ is exact and the pullback of $\eta^g - \eta$ to ∂D is equal to 0.

Proposition 3.3. *The flux homomorphism* $\mathrm{Flux}_{\mathbb{R}}$ *is surjective.*

Proof. Let $\alpha : [0,1] \to \mathbb{R}$ be a non-negative C^∞ function such that $\alpha(x) = 0$ on a neighborhood of $\{0,1\}$ and $\alpha(1/2) = 1$. For $s \in \mathbb{R}$, define the symplectomorphism h_s by $h_s(r,\theta) = (r, \theta + s\alpha(r))$ where $(r,\theta) \in D$ is the polar coordinates. Then we have $\mathrm{Flux}_{\mathbb{R}}(h_1) > 0$ and $\mathrm{Flux}_{\mathbb{R}}(h_s) = s\,\mathrm{Flux}_{\mathbb{R}}(h_1)$, and the surjectivity follows. □

The right-hand side of formula (3.1) also defines a 1-cochain $\tau : G \to \mathbb{R}$:

$$\tau(g) = -\int_\gamma \delta\eta(g). \tag{3.2}$$

Unlike $\mathrm{Flux}_{\mathbb{R}}$, the map τ does depend on the choice of η and γ at the outside of G_{rel}. From now on, we fix η and γ as

$$\eta = (xdy - ydx)/2 \ \text{ and } \gamma : [0,1] \to D; \gamma(t) = (t, 0). \tag{3.3}$$

REMARK 3.4. We can generalize all the discussions after here for arbitrary choices of η satisfying $\omega = d\eta$ and γ connecting the origin and the boundary.

Proposition 3.5. *For $g, h \in G$, the following holds:*

$$-\delta\tau(g,h) = \frac{1}{2}(\varphi\psi(0) - \varphi(0) - \psi(0)), \tag{3.4}$$

where $\varphi, \psi \in \widetilde{\mathrm{Diff}_+(S^1)}$ are lifts of the restrictions $g|_{\partial D}, h|_{\partial D} \in \mathrm{Diff}_+(S^1)$ respectively.

Proof. Minus the coboundary of τ is

$$-\delta\tau(g,h) = \int_{\gamma - h\gamma} \delta\eta(g). \tag{3.5}$$

By the Stokes formula and the exactness of $\delta\eta(g)$, the integration (3.5) depends only on the endpoints of $\gamma - h\gamma$. So we have

$$-\delta\tau(g,h) = \int_{h(x_0)}^{x_0} \delta\eta(g) = -\int_{x_0}^{h(x_0)} \delta\eta(g), \tag{3.6}$$

where $x_0 = \gamma(1) = (1,0)$. Using the polar coordinates (r,θ), we put

$$g(r,\theta) = (g_1(r,\theta), g_2(r,\theta)) \in D.$$

Then the restriction of the integrand $\delta\eta(g)$ to the boundary ∂D becomes $\frac{1}{2}(d\theta - dg_2)$. Since φ is a lift of $g|_{\partial D}(\theta) = g_2(1,\theta) \in \mathrm{Diff}_+(S^1)$, we obtain

$$\begin{aligned}-\delta\tau(g,h) &= -\frac{1}{2}\int_{x_0}^{h(x_0)} d\theta - dg_2 = -\frac{1}{2}\int_0^{\psi(0)} dx - \frac{\partial\varphi}{\partial x}(x)\,dx \\ &= \frac{1}{2}(\varphi\psi(0) - \varphi(0) - \psi(0)).\end{aligned}$$

□

REMARK 3.6. The equality (3.4) implies that the cochain τ is a quasi-morphism, that is, $\delta\tau$ is a bounded function. In fact, the absolute value $\frac{1}{2}|\varphi\psi(0) - \varphi(0) - \psi(0)|$ is bounded by π (see [3] for more details).

Next we show the formulas which are similar to Kotschick-Morita[5, Lemma 6].

Corollary 3.7. *For $g \in G, h \in G_{\mathrm{rel}}$, the following hold:*

i) $\tau(gh) = \tau(g) + \mathrm{Flux}_{\mathbb{R}}(h)$ *and* $\tau(hg) = \tau(g) + \mathrm{Flux}_{\mathbb{R}}(h)$.

ii) *The map* $\mathrm{Flux}_{\mathbb{R}} : G_{\mathrm{rel}} \to \mathbb{R}$ *is a homomorphism satisfying*

$$\mathrm{Flux}_{\mathbb{R}}(ghg^{-1}) = \mathrm{Flux}_{\mathbb{R}}(h).$$

iii) *The kernel of the flux homomorphism* $K = \mathrm{Ker}\,\mathrm{Flux}_{\mathbb{R}}$ *is a normal subgroup of* G.

Proof. As in the Proposition 3.5, we take lifts φ and ψ in $\widetilde{\mathrm{Diff}_+(S^1)}$ of $g|_{\partial D}$ and $h|_{\partial D}$ respectively. Since $h \in G_{\mathrm{rel}}$, we have $\psi = T^n$ for some integer $n \in \mathbb{Z}$, where $T : \mathbb{R} \to \mathbb{R}$ is the translation $T(x) = x + 2\pi$. Thus, we obtain

$$\delta\tau(g,h) = \frac{1}{2}(\varphi(0) - \varphi T^n(0) + T^n(0)) = \frac{1}{2}(\varphi(0) - \varphi(0) - 2n\pi + 2n\pi) = 0$$

and

$$\delta\tau(h,g) = \frac{1}{2}(T^n(0) - T^n\varphi(0) + \varphi(0)) = \frac{1}{2}(2n\pi - \varphi(0) - 2n\pi + \varphi(0)) = 0.$$

On the other hand, we have

$$\delta\tau(g,h) = \mathrm{Flux}_{\mathbb{R}}(h) - \tau(gh) + \tau(g) \quad \text{and} \quad \delta\tau(h,g) = \tau(g) - \tau(hg) + \mathrm{Flux}_{\mathbb{R}}(h),$$

which proves i). The equalities in i) prove ii) and the equality in ii) proves iii). □

According to Corollary 3.7, we have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

where $\mathbb{R}$ is identified with G_{rel}/K.

Proposition 3.8. *The sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1 \tag{3.7}$$

is a central $\mathbb{R}$*-extension.*

Proof. Note that the equality $h_1K = h_2K$ is equivalent to $\mathrm{Flux}_{\mathbb{R}}(h_1) = \mathrm{Flux}_{\mathbb{R}}(h_2)$ for every $h_1, h_2 \in G_{\mathrm{rel}}$. Since $\mathrm{Flux}_{\mathbb{R}}(ghg^{-1}) = \mathrm{Flux}_{\mathbb{R}}(h)$ for $g \in G$ and $h \in G_{\mathrm{rel}}$, we obtain

$$ghK = ghg^{-1}gK = ghg^{-1}K \cdot gK = hK \cdot gK = hgK.$$

The above equality implies that the sequence (3.7) is a central extension. □

DEFINITION 3.9. The central $\mathbb{R}$-extension

$$0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1$$

is called the *flux extension*.

The 1-cochain τ of (3.2) induces a connection cochain $\overline{\tau} \in C^1(G/K; \mathbb{R})$ of the flux extension because of the Corollary 3.7 i). Thus the formula (3.4) gives the following Euler cocycle of flux extension

$$-\delta\overline{\tau}(gK, hK) = \frac{1}{2}(\varphi\psi(0) - \varphi(0) - \psi(0)),$$

where φ, ψ are defined as in Proposition 3.5 for $g, h \in G$. Using the formula (2.1) with $c = 0$, we obtain

$$-\delta\overline{\tau}(gK, hK) = \pi\chi(\mu, \nu),$$

where $\mu = g|_{\partial D}$ and $\nu = h|_{\partial D}$. Consequently we have the following:

Theorem 3.10. *The class $e(G/K)$ is equal to $e_{\mathbb{R}}$ up to constant multiple. More precisely, the following holds:*

$$e(G/K) = \pi e_{\mathbb{R}}.$$

4. Ismagilov-Losik-Michor's cocycle

Let M be a connected symplectic manifold with exact symplectic form ω. Assume that the first Betti number of M is 0. Then, there is a 2-cocycle C_{η,x_0} in $C^2(\mathrm{Symp}(M); \mathbb{R})$ introduced by Ismagilov, Losik, and Michor [4], which we call the ILM cocycle. For $x_0 \in M$ and $\eta \in \Omega^1(M)$ satisfying $d\eta = \omega$, the ILM cocycle is defined as

$$C_{\eta,x_0}(g, h) = -\int_{x_0}^{h(x_0)} \delta\eta(g) = \int_{x_0}^{h(x_0)} \eta^g - \eta \tag{4.1}$$

for $g, h \in \mathrm{Symp}(M)$. They proved that the cohomology class $[C_{\eta,x_0}]$ is independent of the choice of $x_0 \in M$ and the potential $\eta \in \Omega^1(M)$.

REMARK 4.1. Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half-plane with the symplectic (area) form $dx \wedge dy/y^2$. Then the group $PSL(2, \mathbb{R})$ is considered as a subgroup of $\mathrm{Symp}(\mathbb{H})$

and the restriction of the ILM cocycle to $PSL(2,\mathbb{R})$ is cohomologous to the area 2-cocycle which is defined as the area of geodesic triangles (see [4]).

The ILM cocycle corresponds to the symplectic form ω in the double complex $(C^\bullet(\mathrm{Symp}(M);\Omega^\bullet(M)),\delta,d)$ as follows: Considering a real number as a constant function on M, there is an inclusion

$$i : C^\bullet(\mathrm{Symp}(M);\mathbb{R}) \hookrightarrow C^\bullet(\mathrm{Symp}(M);\Omega^0(M)).$$

There exists a function $\mathcal{K}_{\eta,x_0}(g)$ satisfying

$$d\mathcal{K}_{\eta,x_0}(g) = \delta\eta(g) \quad \text{and} \quad \mathcal{K}_{\eta,x_0}(g)(x_0) = 0$$

because of $H^1(M;\mathbb{R}) = 0$. Since the manifold M is connected, this function $\mathcal{K}_{\eta,x_0}(g)$ is uniquely determined for each $g \in \mathrm{Symp}(M)$. Then we obtain the 1-cochain $\mathcal{K}_{\eta,x_0} \in C^1(\mathrm{Symp}(M);\Omega^0(M))$. A straightforward calculation shows that $\delta\mathcal{K}_{\eta,x_0} = -i(C_{\eta,x_0})$ (see Proposition 2.3 of [2]). Then these cochains $\omega, \eta, \mathcal{K}_{\eta,x_0}$ and C_{η,x_0} are connected via the following diagram:

$$\begin{tikzcd}
\omega \arrow[r, mapsto, "\delta"] & 0 & \\
\eta \arrow[u, mapsto, "d"] \arrow[r, mapsto, "\delta"] & \bullet \arrow[u, mapsto, "d"] \arrow[r, mapsto, "\delta"] & 0 \\
& \mathcal{K}_{\eta,x_0} \arrow[u, mapsto, "d"] \arrow[r, mapsto, "\delta"] & \bullet \arrow[u, mapsto, "d"] \\
& & -C_{\eta,x_0} \arrow[u, mapsto, "i"]
\end{tikzcd} \tag{4.2}$$

REMARK 4.2. We consider the differential forms $\Omega^\bullet(M)$ as a right $\mathrm{Symp}(M)$-module by pullback.

REMARK 4.3. In terms of $\mathcal{K}_{\eta,x_0}$, the flux homomorphism $\mathrm{Flux}_{\mathbb{R}}$ is expressed by

$$\mathrm{Flux}_{\mathbb{R}}(g) = \mathcal{K}_{\eta,x_0}(g)(o).$$

Let us return to the disk case. Take η and γ as (3.3) and put $x_0 = \gamma(1) = (1,0)$. We consider the ILM cocycle C_{η,x_0} on $H = \mathrm{Symp}(D)$. Let $i : G \hookrightarrow H$ be the inclusion. By the equality (3.6), the pullback of C_{η,x_0} by i is a coboundary, that is,

$$-\delta\tau = i^* C_{\eta,x_0} \in C^2(G;\mathbb{R}). \tag{4.3}$$

So we have proved

Theorem 4.4. *Take η, x_0 as above. On the sequence*

$$1 \longrightarrow G_{\mathrm{rel}} \longrightarrow G \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

the pullback $i^* C_{\eta,x_0}$ of the ILM cocycle is basic and its basic cocycle represents $\pi e_{\mathbb{R}}$.

Remark 4.5. For arbitrary η and $x_0 \in \partial D$, we can prove Theorem 4.4 (and also Theorem 5.7 below) by taking the path γ from the origin to x_0.

Corollary 4.6. *The pullback $i^* C_{\eta,x_0}$ is a bounded cocycle.*

Proof. Since τ is a quasi-morphism, the boundedness follows. □

5. The Calabi invariant and extension

In this section, we investigate the relation between the ILM cocycle and the Calabi invariant. First we summarize the results in [7], which are needed to us. Let us recall the Calabi invariant and the Calabi extension. Denote by H_{rel} the group consisting of the relative symplectomorphisms on D, that is, of the symplectomorphisms on D whose restrictions to the boundary ∂D are equal to the identity $\mathrm{id}_{\partial D}$:

$$H_{\mathrm{rel}} = \{g \in H \mid g|_{\partial D} = \mathrm{id}\}.$$

Then there is an exact sequence

$$1 \longrightarrow H_{\mathrm{rel}} \longrightarrow H \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1.$$

On the group H_{rel}, the homomorphism $\mathrm{Cal} : H_{\mathrm{rel}} \to \mathbb{R}$ is defined by

$$\mathrm{Cal}(g) = \int_D \eta \cup \delta\eta(g) = \int_D \eta^g \wedge (\eta - \eta^g), \tag{5.1}$$

where the symbol $\cup$ is the cup product of group cochains defined as follows: for $\alpha \in C^p(G; \Omega^r(D)), \beta \in C^q(G; \Omega^s(D))$ and $g_1, \dots, g_{p+q} \in G$,

$$\alpha \cup \beta(g_1, \dots, g_{p+q}) = \alpha(g_1, \dots, g_p)^{g_{p+1} \cdots g_{p+q}} \wedge \beta(g_{p+1}, \dots, g_{p+q}).$$

This homomorphism is called the Calabi invariant (see [6] for details). It is known that the Calabi invariant is a surjective homomorphism and $L = \mathrm{Ker}\,\mathrm{Cal}$ is a normal subgroup of H. Dividing the above exact sequence by L, we have the following central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow H/L \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1$$

called the Calabi extension.

Remark 5.1. There are no inclusion relation between $L \cap G$ and $K = \mathrm{Ker}\,\mathrm{Flux}_{\mathbb{R}}$.

Next we consider the resulting class $e(H/L)$ of the Calabi extension. The right-hand side of the formula (5.1) also defines the 1-cochain $\tau_0 : H \to \mathbb{R}$. The 1-cochain $\tau_0 \in C^1(H; \mathbb{R})$ induces a connection cochain

$$\overline{\tau_0} \in C^1(H/L; \mathbb{R})$$

of the Calabi extension. The curvature $\delta\overline{\tau_0}$ is basic and this basic cocycle, denoted by the same letter $\delta\overline{\tau_0}$, coincides with $-\pi^2\chi - \pi^2/2$. Thus we have $e(H/L) = \pi^2 e_{\mathbb{R}}$ (see Moriyoshi [7, Theorem 2]).

For η, x_0 in Theorem 4.4, the ILM cocycle C_{η,x_0} gives a curvature of the Calabi extension. Define a 1-cochain $\kappa \in C^1(H; \mathbb{R})$ by

$$\kappa = \int_{\partial D} \mathcal{K}_{\eta,x_0} \cup \eta.$$

Remark 5.2. This 1-cochain κ is a non-zero function.

Proposition 5.3. *Set $\tau' = \tau_0 + \kappa \in C^1(H;\mathbb{R})$. The following holds:*

$$-\delta\tau' = \pi C_{\eta,x_0}.$$

Proof. For $g \in H$, we have

$$\begin{aligned} \tau_0(g) &= \int_D \eta^g \wedge \eta = -\int_D \delta\eta(g) \wedge \eta = -\int_D (d\mathcal{K}_{\eta,x_0}(g)) \wedge \eta \\ &= \int_D \mathcal{K}_{\eta,x_0}(g)\omega - \int_{\partial D} \mathcal{K}_{\eta,x_0}(g)\eta = \left(\int_D \mathcal{K}_{\eta,x_0} \cup \omega - \kappa\right)(g), \end{aligned} \tag{5.2}$$

where the fourth equality follows from the Stokes formula. Recalling that $\delta\mathcal{K}_{\eta,x_0} = -C_{\eta,x_0}$, we get

$$\delta\tau_0 = \int_D (-C_{\eta,x_0}) \cup \omega - \delta\kappa = -\pi C_{\eta,x_0} - \delta\kappa.$$

This proves $-\delta\tau' = \pi C_{\eta,x_0}$. □

Remark 5.4. Combining equality (4.3) and Proposition 5.3, we have $\delta(\pi\tau - \tau'|_G) = 0$. So the function $\pi\tau - \tau'|_G : G \to \mathbb{R}$ is a homomorphism. Furthermore, this homomorphism is surjective.

Next, we show τ' also gives rise to a connection cochain.

Lemma 5.5. *For $g \in H, h \in H_{\mathrm{rel}}$, we have the following:*

i) $\kappa(h) = 0$*, and*

ii) $\kappa(gh) = \kappa(g) = \kappa(hg)$.

Proof. i) The 1-cochain $\mathcal{K}_{\eta,x_0}$ satisfies

$$d\mathcal{K}_{\eta,x_0}(h) = \delta\eta(h) \quad \text{and} \quad \mathcal{K}_{\eta,x_0}(h)(x_0) = 0 \tag{5.3}$$

by the definition of $\mathcal{K}_{\eta,x_0}$. The restriction of closed form $\delta\eta(h)$ to ∂D is equal to 0 because the restriction $h|_{\partial D}$ is the identity. There exists a 0-form $f : D \to \mathbb{R}$ such that

$$\delta\eta(h) = df \quad \text{and} \quad f|_{\partial D} = 0 \tag{5.4}$$

because of $H^1(D, \partial D; \mathbb{R}) = 0$. From the equalities in (5.3) and (5.4), we deduce

$$\mathcal{K}_{\eta,x_0}(h) - f = c,$$

where c is a constant in $\mathbb{R}$. Evaluating $\mathcal{K}_{\eta,x_0}(h) - f$ at x_0, we have $c = 0$. So we obtain

$$\kappa(h) = \int_{\partial D} \mathcal{K}_{\eta,x_0} \cup \eta(h) = \int_{\partial D} f \cdot \eta = 0.$$

ii) Note that

$$d(\mathcal{K}_{\eta,x_0}(gh) - \mathcal{K}_{\eta,x_0}(g)) = \eta^g - \eta^{gh} = d\mathcal{K}_{\eta^g,x_0}(h)$$

and $(\mathcal{K}_{\eta,x_0}(gh)-\mathcal{K}_{\eta,x_0}(g))(x_0) = 0 = \mathcal{K}_{\eta^g,x_0}(h)(x_0)$. By the same argument in i), the restriction $\mathcal{K}_{\eta^g,x_0}(h)|_{\partial D}$ is equal to 0. So we have

$$\mathcal{K}_{\eta,x_0}(gh)|_{\partial D} = \mathcal{K}_{\eta,x_0}(g)|_{\partial D}$$

and this equality induces $\kappa(gh) = \kappa(g)$. Applying the similar argument for

$$d(\mathcal{K}_{\eta,x_0}(hg) - \mathcal{K}_{\eta,x_0}(g)) = \eta^g - \eta^{hg} = d(\mathcal{K}_{\eta,x_0}(h)^g),$$

we obtain $\kappa(hg) = \kappa(g)$. □

Lemma 5.6. *For $g \in H, h \in H_{\mathrm{rel}}$, the following hold:*

$$\tau'(gh) = \tau'(g) + \mathrm{Cal}(h) = \tau'(hg). \tag{5.5}$$

Proof. The 1-cochain τ_0 satisfies

$$\tau_0(gh) = \tau_0(g) + \mathrm{Cal}(h) = \tau_0(hg),$$

where $g \in H$ and $h \in H_{\mathrm{rel}}$ (see [7, Proposition 4]). Combining the above equalities and Lemma 5.5, we obtain the equalities (5.5). □

From Lemma 5.6, the cochain τ' induces the connection cochain $\overline{\tau'} \in C^1(H/L;\mathbb{R})$ of the Calabi extension. So $-\delta\overline{\tau'}$ gives the Euler class $e(H/L) \in H^2(\mathrm{Diff}_+(S^1);\mathbb{R})$. Recalling $e(H/L) = \pi^2 e_{\mathbb{R}}$, we obtain

Theorem 5.7. *For η, x_0 in* Theorem 4.4*, The ILM cocycle C_{η,x_0} is basic with respect to the exact sequence*

$$1 \longrightarrow H_{\mathrm{rel}} \longrightarrow H \longrightarrow \mathrm{Diff}_+(S^1) \longrightarrow 1$$

and the cohomology class of basic cocycle is equal to $\pi e_{\mathbb{R}}$.

Remark 5.8. The integration in definition of ILM cocycle depends only on the restrictions of g, h to ∂D. Hence, from Corollary 4.6, the ILM cocycle C_{η,x_0} is also a bounded cocycle. Since $\delta\tau'$ is equal to $-\pi C_{\eta,x_0}$, the cochain τ' gives rise to a quasi-morphism on H. In [7], it is proved that the cochain τ_0 is a quasi-morphism. Thus the cochain κ is also a quasi-morphism on H because of $\tau' = \tau_0 + \kappa$.

References

[1] K.S. Brown: Cohomology of Groups, Graduate Texts in Mathematics **87**, Springer-Verlag, New York-Berlin, 1982.

[2] Ś. Gal and J. Kędra: *A two-cocycle on the group of symplectic diffeomorphisms*, Math. Z. **271** (2012), 693–706.

[3] E. Ghys: *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), 329–407.

[4] R.S. Ismagilov, M. Losik, and P.W. Michor: *A 2-cocycle on a symplectomorphism group*, Mosc. Math. J. **6** (2006), 307–315.

[5] D. Kotschick and S. Morita: *Signatures of foliated surface bundles and the symplectomorphism groups of surfaces*, Topology **44** (2005), 131–149.

[6] D. McDuff and D. Salamon: Introduction to Symplectic Topology, Oxford mathematical monographs, Clarendon Press, Oxford University Press, New York, 1998.
[7] H. Moriyoshi: *The Calabi invariant and central extensions of diffeomorphism groups*; in Geometry and topology of manifolds, 283–297, Springer Proc. Math. Stat. **154**, Springer, Tokyo, 2016.

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